### Necessary Conditions for Hypergraph Transitivity

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**Goal**
Assumption: \((V, E)\) is a \(d\)-regular, \(k\)-uniform hypergraph with \(n\) vertices and \(m\) edges.

Question: What necessary conditions must the hypergraph satisfy to be vertex-, edge-, or flag-transitive?

### Basic Definitions
- A **hypergraph** \((V, E)\) is a structure consisting of a **vertex set** \(V\) and an **edge set** \(E\), where each edge is a set of vertices.
- A **flag** of a hypergraph is a pair \((v, e)\), where \(v\) is a vertex contained in the edge \(e\).
- An **automorphism** of a hypergraph is a permutation of \(V\) that also maps edges to edges.
- A hypergraph is **vertex-transitive** if any vertex can be sent to any other vertex by an automorphism.
- Edge-transitive and flag-transitive are defined similarly.
- \((V, E)\) is **\(k\)**-uniform if each edge contains \(k\) vertices, and it is **\(d\)**-regular if each vertex is contained in \(d\) edges.

### Dual
- The dual of a hypergraph is obtained by swapping the edges and vertices.
- A hypergraph is vertex-transitive iff its dual is edge-transitive, and vice versa.
- This allows vertex-transitivity conditions to be converted to edge-transitivity conditions, and vice-versa.

### Specific Conditions
- **Definition**: Define \(E_{i,j}\) to be the set of all \(i\)-vertex edges contained in exactly \(j\) edges of \(E\). Let \(m_{i,j} = |E_{i,j}|\)

**General Equivalences for all Hypergraphs:**
\[
\sum_{j=0}^{\infty} m_{i,j} = \binom{n}{i} \quad \sum_{j=1}^{\infty} jm_{i,j} = \binom{k}{i}
\]

**Necessary Conditions for Vertex-transitivity:**
- \(E_{i,j}\) is vertex transitive.
- Each vertex is contained in the same number of edges of \(E_{i,j}\) that contain each vertex.

**Necessary Conditions for Edge-transitivity:**
- Each edge contains the same number of edges \(E_{i,j}\) that contain each edge.

**Necessary Conditions for Flag-transitivity:**
- For each flag \((v,e)\), there are the same number of edges \(\omega \in E_{i,j}\) such that \(v \in \omega \subseteq e\)
- For each edge \(e\), the set of all edges in \(E_{i,j}\) that are contained in \(e\) forms a vertex-transitive edge set for \(e\).

### Example
- Each colored triangle is a 3-vertex edge.
- \((V, E)\) is vertex- and edge-transitive because rotations and reflections are automorphisms.
- \((V, E)\) is **not** flag-transitive, because same-color flags are incident on more edges in \(E_{2,2}\) than different-color flags.

### Conclusions
- For a hypergraph \((V, E)\) to be vertex-transitive, the vertex set must be regular in any hypergraph that is preserved by the automorphisms of \((V, E)\).
- Similar statements hold for edge- and flag-transitive hypergraphs.
- An example of an edge set that is preserved by automorphisms of \(E\) is the set of all \(i\)-vertex edges that are contained in \(j\) edges of \(E\).
- Edge sets that are preserved by automorphisms of \(E\) can be used to generate more such edge sets by restricting one to a subset that is regular in another.
- Edge-transitivity conditions can be converted to vertex-transitivity conditions and vice versa with the dual.
- More general conditions can be described using first order logic and model theory.
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