Necessary Conditions for Hypergraph Transitivity

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Mathematics

Goal

Assumption: (V, E) is a d-regular, k-uniform hypergraph with n vertices and m edges.

Question: What necessary conditions must the hypergraph satisfy to be vertex-, edge-, or flag-transitive?

Basic Definitions

- A <u>hypergraph</u> (V, E) is a structure consisting of a <u>vertex set</u> V and an <u>edge set</u> E, where each edge is a set of vertices.
- A <u>flag</u> of a hypergraph is a pair (v, e), where v is a vertex contained in the edge e.
- An <u>automorphism</u> of a hypergraph is a permutation of V that also maps edges to edges.
- A hypergraph is <u>vertex-transitive</u> if any vertex can be sent to any other vertex by an automorphism.
- <u>Edge-transitive</u> and <u>flag-transitive</u> are defined similarly.
- (V, E) is <u>k-uniform</u> if each edge contains k vertices, and it is <u>d-regular</u> if each vertex is contained in d edges.

Dual

- The dual of a hypergraph is obtained by swapping the edges and vertices
- A hypergraph is vertex-transitive iff its dual is edge-transitive, and vice versa.
- This allows vertex-transitivity conditions to be converted to edge-transitivity conditions, and vice-versa.

Specific Conditions

<u>Definition</u>**:** Define $E_{i,j}$ to be the set of all i-vertex edges contained in exactly j edges of E. Let $m_{i,j} = |E_{i,j}|$

<u>General Equalities for all Hypergraphs</u>:

$\sum_{j=0}^{\infty} m_{i,j} = \binom{n}{i}$ $\qquad$ $\sum_{j=1}^{\infty} j m_{i,j} = m\binom{k}{i}$

<u>Necessary Conditions for Vertex-transitivity</u>:

- $E_{i,j}$ is vertex transitive.
- Each vertex is contained in the same number of edges of $E_{i,j}$

$$i m_{i,j} = n d_{i,j}$$

Where $d_{i,j}$ is the number of edges in $E_{i,j}$ that contain each vertex.

<u>Necessary Conditions for Edge-transitivity</u>:

- Each edge contains the same number of edges of $E_{i,j}$

$$m l_{i,j} = j m_{i,j}$$

Where $m_{i,j}$ is the number of edges in $E_{i,j}$ containing each edge.

<u>Necessary Conditions for Flag-transitivity</u>:

- For each flag (v, e), there are the same number of edges $\omega \in E_{i,j}$ such that $v \in \omega \subseteq e$
- For each edge e, the set of all edges in $E_{i,j}$ that are contained in e forms a vertex-transitive edge set for e.

General Conditions

<u>Definition</u>: If x is a vertex, edge, or flag, and e is an edge, then x and e are <u>incident</u> if

- $x \in e$, if x is a vertex
- $x \subseteq e$ or $e \subseteq x$, if x is an edge
- $\dot{x} \in e \subseteq \overline{x}$, if $x = (\dot{x}, \overline{x})$ is a flag

<u>Definition</u>: If S is a set of vertices, edges, or flags, and E is an edge set, S is <u>regular</u> in E if each element of S is incident on the same number of edges in E.

<u>Definition</u>: If E and F are edge sets, then $E \leq F$ if every automorphism of E is an automorphism of F.

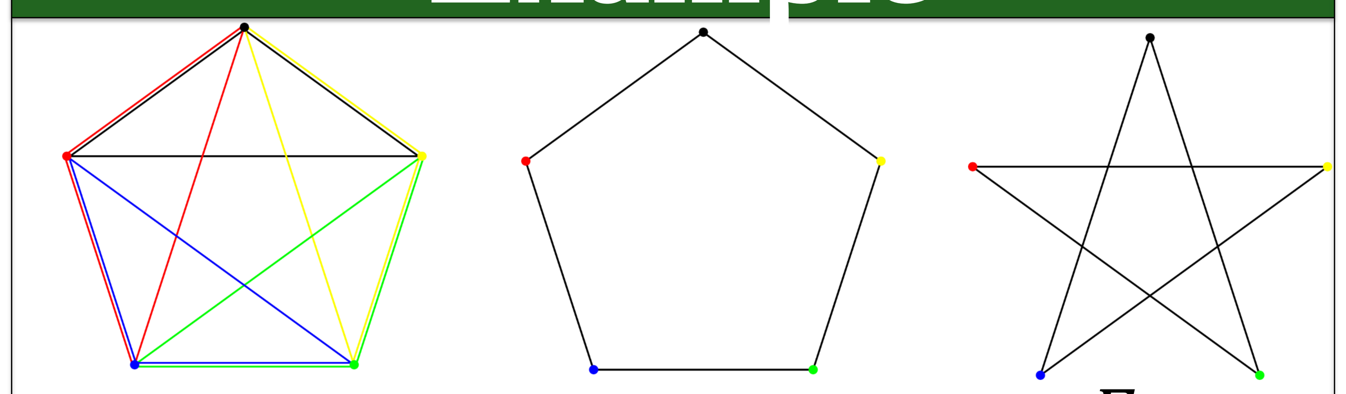
<u>Theorem</u>: If (V, E) is (vertex/edge/flag)-transitive and $E \leq F$ then the (vertex/edge/flag)-set of (V, E) is regular in F.

Note that $E \leq E_{i,j}$, so above conditions are a special case of this. More conditions can be generated by generating more edge sets $\geq E$. This can be done with the following theorem.

<u>Theorem</u>: If $E_0 \leq E, F$ and E_i^F is the set of all edges in E that are incident on exactly i edges in F, then $E_0 \leq E_i^F$

The theorem above can be applied recursively, e.g. the set of all edges in $E_{2,3}$ that are incident on 2 edges in $E_{4,2}$ is $\geq E$.

Example



E $\qquad$ $E_{2,2}$ $\qquad$ $E_{2,1}$

- Each colored triangle is a 3-vertex edge.
- (V, E) is vertex- and edge-transitive because rotations and reflections are automorphisms.
- (V, E) is NOT flag-transitive, because same-color flags are incident on more edges in $E_{2,2}$ than different-color flags.

Conclusions

- For a hypergraph (V, E) to be vertex-transitive, the vertex set must be regular in any hypergraph that is preserved by the automorphisms of (V, E)
- Similar statements hold for edge- and flag-transitive hypergraphs.
- An example of an edge set that is preserved by automorphisms of E is the set of all i-vertex edges that are contained in j edges of E.
- Edge sets that are preserved by automorphisms of E can be used to generate more such edge sets by restricting one to a subset that is regular in another.
- Edge-transitivity conditions can be converted to vertex-transitivity conditions and vice versa with the dual.
- More general conditions can be described using first order logic and model theory.
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