POD Approximations and Errors

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Mathematics

POD and Error Approximations

- **Proper Orthogonal Decomposition** is a model order reduction technique for PDEs
- Provides modes from simulation data and a projection is used on the modes to obtain the reduced order model (ROM)
- Optimal method to "compress" data
- Extension of singular value decomposition (SVD) for matrices
- POD depends on the Hilbert space structure of the data space
- The POD changes with the norm
- Errors of these models are of concern.
- Changing norms, i.e., the way to measure the error, can change errors of approximation

Basic Problem

Let w be the solution of the PDE and w_r be the solution of the ROM. The error is

$$w - w_r = \ (w - \pi_r w) + (\pi_r w - w_r),$$

where π_r is a projection. How does $w - \pi_r w$ behave in the POD setting?

Background

- Linear operator $L: X \to Y$, where X and Y are Hilbert spaces
- Two sets of data: $\{w_k\} \subset X$ and $\{Lw_k\} \subset Y$
- POD operator $K: S \to X$ with SVD given by $Kf = \sum_{j=1}^{m} \sigma_j (f, f_j)_S \varphi_j$
- Two projections: $\Pi_r^X: X \to X$ is the orthogonal projection onto $span\{\varphi_k\}$ and $\Pi_r^Y: Y \to Y$ is a projection onto $span\{L\varphi_k\}$
- Known error for the data $\{w_j\} \subset X$:

$$\sum_{j=1}^{m} \left\| w_j - \Pi_r^X w_j \right\|_X^2 = \sum_{k>r} \sigma_k^2$$

Main Results

Theorem 1: If $\{w_k\}$ is in the domain of L and $\sigma_r > 0$, data approximation errors are given by

$$\sum_{j=1}^{m} \left\| Lw_j - L\Pi_r^X w_j \right\|_Y^2 = \sum_{k>r} \sigma_k^2 \| L\varphi_k \|_Y^2$$

$$\sum_{j=1}^{m} \left\| Lw_j - \Pi_r^Y Lw_j \right\|_Y^2 = \sum_{k>r} \sigma_k^2 \left\| L\varphi_k - \Pi_r^Y L\varphi_k \right\|_Y^2$$

And

$$\sum_{j=1}^{m} \left\| w_j - L^{-1}\Pi_r^Y Lw_j \right\|_X^2 = \sum_{k>r} \sigma_k^2 \left\| \varphi_k - L^{-1}\Pi_r^Y L\varphi_k \right\|_X^2$$

Further, there are conditions which guarantee the convergence of these norms.

Theorem 2: Assume L is bounded and $\{\Pi_r^Y\}$ is uniformly bounded in operator norm.

1. If $y \in \mathcal{R}(L)$, then $\Pi_r^Y y \to y$ as r increases.
2. In addition, if $\mathcal{R}(L)$ is dense in Y, then $\Pi_r^Y y \to y$ for all $y \in Y$.

Numerical Example

Consider a nerve impulse transmission model: the 1D FitzHugh-Nagumo system given by

$$\frac{\partial v(t,x)}{\partial t} = \mu \frac{\partial^2 v(t,x)}{\partial x^2} - \frac{1}{\mu} w(t,x) + \frac{1}{\mu} f(v) + \frac{c}{\mu}, \qquad 0 < x < 1$$

$$\frac{\partial w(t,x)}{\partial t} = bv(t,x) - \gamma w(t,x) + c, \qquad 0 < x < 1$$

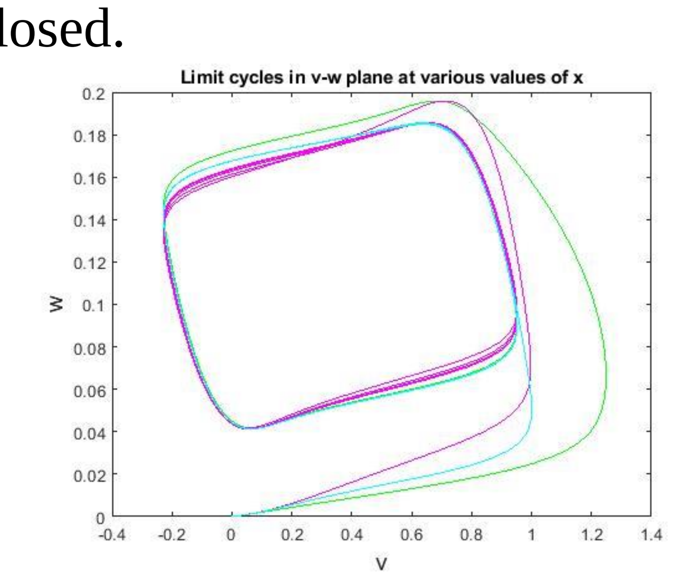
where $f(v) = v(v - 0.1)(1 - v)$, $\mu = 0.015, b = 0.5, \gamma = 2, c = 0.05$, the boundary conditions are $\mathrm{v_x(t,0)} = -50000\mathrm{t}^3\mathrm{e}^{-15\mathrm{t}}, \mathrm{v_x(t,1)} = 0$, and the initial conditions are zero.

The Hilbert spaces are $X = Y = L^2(0,1) \ x \ L^2(0,1)$ with the usual inner product.

Define the operator $L: X \to Y$ by

$$L\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} \partial_x v \\ \partial_x w \end{pmatrix}$$

Note that L is unbounded and closed.



Formula	Norm	Actual Error	Error Formula	Difference
$w_j - \Pi_r^X w_j$	X	6.2755e-05	6.2792e-05	3.7584e-08
$Lw_j - L\Pi_r^X w_j$	Y	2.1584e-01	2.1593e-01	9.1863e-05
$Lw_j - \Pi_r^Y Lw_j$	Y	9.8536e-03	9.8541e-03	4.7712e-07

Table 1: Errors for r = 4

Formula	Norm	Actual Error	Error Formula	Difference
$w_j - \Pi_r^X w_j$	X	4.145e-08	4.148e-08	3.366e-11
$Lw_j - L\Pi_r^X w_j$	Y	2.253e-04	2.254e-04	5.214e-08
$Lw_j - \Pi_r^Y Lw_j$	Y	1.266e-05	1.266e-05	3.515e-09

Table 2: Errors for r = 12

More Results

- Main results can be extended to the case of continuous time data.
- Results hold for a non-orthogonal projection Π_r^Y, in particular we can use a Ritz projection.
- The generalized framework allows for consideration of spaces with semi-norms.
- Results include properties of various POD projections including pointwise convergence
- Before these results, inverse inequalities were used to bound the errors. Now we have exact error formulas that do not require inverse estimates.
- These results refine existing data approximation results and provide new extensions.

Future Research

- When are the projections involving the inverse of the operator uniformly bounded?
- Consider new ways of including difference quotients in POD ROMS with the goal to improve accuracy and efficiency
- Explore pointwise in time error bounds for POD ROMs

References

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John R. Singler. New POD Error Expressions, Error Bounds, and Asymptotic Results for Reduced Order Models of Parabolic PDEs. SIAM Journal on Numerical Analysis, vol. 52, no. 2, 2014, pp.852-876.

Acknowledgments

I would like to thank my advisor, Dr. John Singler, for helping me start this project and the Graduate Studies Office for funding my research through the CDF program.