A Case of the Thanksgiving Lemma Conjecture

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Mathematics

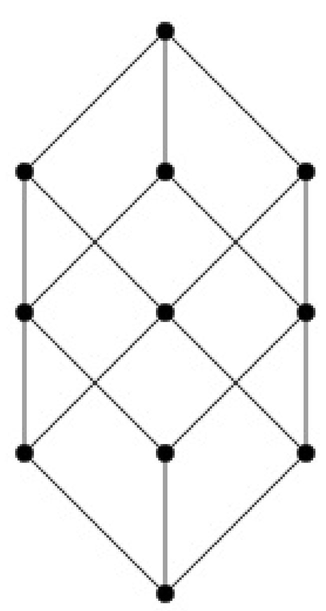
Goals of Research

Goal: Given minimal-representor G-set **A** consisting of the 2-D vector space over the n²-element field, where $n = p^k + 1$, whose fundamental operations are affine invertible functions, and an M_n lattice isomorphic to **A**, prove that if $(x, y) \in \theta(u, v)$ has a minimal "witnessing chain" of length 3, where $u \neq v$, then $(u, v) \in \theta(x, v)$.

Poset & Lattice Def.

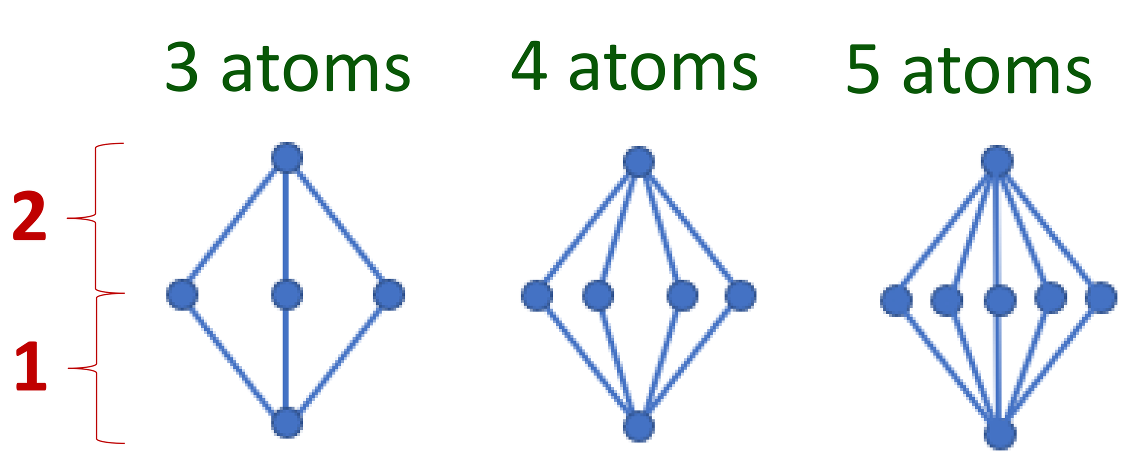
A **poset** is a set **L** with a partial ordered defined on **L**.

Poset **L** is a **lattice** iff for $a, b \in$ **L**, $\sup\{a, b\} \in$ **L** and $\inf\{a, b\} \in$ **L**.



M_n Lattice Def.

An $\boldsymbol{M_n}$ **lattice** is a lattice of height 2 and n atoms.



For the **TLC**, we consider n of the form $p^k + 1$.

"Witnessing Chain" Definition

A **"witnessing chain"** from (a, b) to (c, d) is a sequence of ordered pairs of **A**, each obtained by using one of the valid operations, that shows that (c, d) is in the principle congruence of (a, b).

Valid operations include:

- Symmetric operator application: $(a, b) \to_\sigma (b, a)$
- Transitive operator application: $(a, b) \ldots (b, c) \to_\tau (a, c)$
- Function application: $(a, b) \to_f f(a, b) = (f(a), f(b))$

A **minimal "witnessing chain"** is the shortest "witnessing chain" from (a, b) to (c, d). The below "witnessing chain" has length 3.

$$(a, f(a)) \xrightarrow{f(a, f(a))} \big(f(a), f(f(a))\big) \xrightarrow{\tau} \big(a, f(f(a))\big)$$

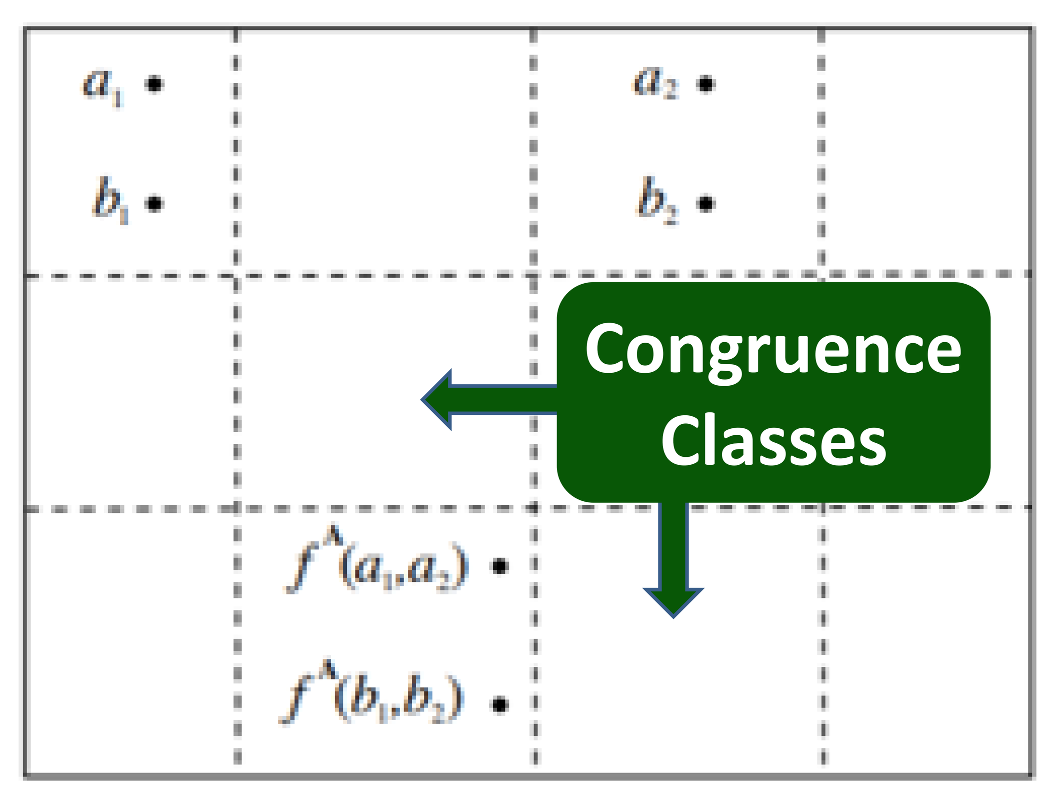
We show that if the above "witnessing chain" is minimal, we can find g such that the below "witnessing chain" exists.

$$(a, f(a)) \xleftarrow[g\left(a, f(f(a))\right)]{} \big(a, f(f(a))\big)$$

Congruence Definitions

For **A** an algebra of type F and $\theta \in \mathrm{Eq}(A)$, θ is a **congruence** on **A** if for each n-ary function $f \in F$ and elements $a_i, b_i \in A$, if $a_i \theta b_i$ holds for $1 \leq i \leq n$, then $f^{\mathbf{A}}(a_1, \ldots, a_n) \theta f^{\mathbf{A}}(b_1, \ldots, b_n)$ holds.

If $\theta(a_1, \ldots, a_n)$ is the congruence generated by $\{\langle a_i, a_j \rangle : 1 \leq i, j \leq n\}$, then $\theta(a_1, a_2)$ is called the **principal congruence** on **A**, i.e. $\theta(a_1, \ldots, a_n)$ is the smallest congruence such that $a_1, \ldots, a_n$ are in the same equivalence class.



Affine Invertible Def.

Affine invertible functions represent vector-valued functions of the form $f(x_1, \ldots, x_n) = A_1 x_1 + \cdots + A_n x_n + b$ where $A_i \neq 0$.

In this case of the TLC, these are polynomials of degree 1 with coefficients in the n²-element field F_{n^2}.

Conclusion

- Defined key elements: **principal congruence** and **"witnessing chain"**
- Proved special case of the **Thanksgiving Lemma Conjecture**: Assume **A** is a minimal-representor G-set consisting of the 2-D vector space over the n²-element field, where $n = p^k + 1$, whose fundamental operations are affine invertible functions and an M_n lattice isomorphic to **A**, if $(x, y) \in \theta(u, v)$ has a minimal "witness" of length 3, where $u \neq v$, $(u, v) \in \theta(x, v)$.
- **Future goals**: 1) Determine method that undoes transitivity for any minimal "witnessing chain," and 2) use this to prove the TLC.
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